Lecturered By Dr. H.K. Jadav
Deptt. of Mathematics
Marwari College, Darbhanga.
B.Sc. part - I (Sub) Date- 13-04-2020

Set Theory.

Q.1 Define equivalence relation and show that the relation '<' in the set of integers is not an equivalence relation.

Soln. Equivalence Relation:→
A relation r in a Set X is said to be an "equivalence relation in X" if r is reflexive, symmetric and Transitive.

Proof: Let I be the set of integers and a relation R in I be defined by $a^b = b^a$, $a, b \in I$.

Then, we have,

(I) $a^a = a^a$, $\forall a \in I$

Thus, relation R is reflexive

(II) $a^b = b^a \Rightarrow b^a = a^b$, $\forall a, b \in I$

Thus, relation R is symmetric

(III) $a^b = b^a$, $b^c = c^b$ do not imply $a^c = c^a$

Thus, relation R is not transitive.

Hence, the relation '<' in the set of integers is not an equivalence relation.

Q.9. State and prove fundamental theorem on equivalence relation.

Soln. Statement:→ An equivalence relation R in a non-empty set S determines a partition of S and conversely a partition of S defines an equivalence relation in S.

Proof: Since R is an equivalence relation in a set S we can have equivalence classes S_a or $[a]$ such that

$$[a] = \{x : x \in S \text{ and } xRa\}$$

Let A be the set of equivalence classes of S with respect to R

$\therefore A = \{[a] : a \in S\}$.

Now we have to prove that this A is a partition of set S for which we have to prove that

(i) all sets of A like $[a]$ are non-empty

(ii) $\cup [a] = S$, and (iii) $a \in S$ sets of A are pairwise disjoints.

I Since R is an equivalence relation which is reflexive.

$\therefore \forall a \in S$, we have aRa.

Hence, $a \in [a]$ and thus $[a]$ is non-empty set.

II Every element a of S is an element of the equivalence class $[a]$ in A and S is the union of all equivalence classes. i.e. $S = \bigcup_{a \in S} [a]$

III Hence, if $[a]$ and $[b]$ are two equivalence classes then ~~their~~ either $[a] = [b]$ or $[a] \cap [b] = \phi$.

B.Sc- part-I (Sub) Set Theory Date - 18-04-2020

Rest part of Q.2:

Hence, A is a partition of set S. Thus we see that an equivalence relation R in S decomposes the set S into equivalence classes, any two of which are either equal or mutually disjoint. (B.Sc. part-I (Sub) End]

Lecturered By Dr H.K. Yadav
Deptt. of Mathematics
Marwari College, Darbhanga.
B.Sc-part-I(H), paper – I, Date-18-04-2020

Set theory

Q.1. Define and explain the terms:
(I) Numerically equivalent sets (II) Finite sets
(III) Denumerable sets (IV) Countable sets (V) uncountable sets
illustrate them with suitable examples.

Soln: I Numerically equivalent sets:→ If there exists a mapping $f: A \to B$, which is one-one onto, then the two sets A and B are said to be numerically equivalent and written as $A \sim B$.

$A \sim$ is read as 'A wiggle B',

For example; Let $f: A \to B$, where $A = \{1,2,3\}$ and $\{2,4,6\}$.
Here, $f(x) = 2x$, $\forall x \in A$ and one-one correspondence exists between A and B. Also f is an onto mapping, hence the sets A and B are equivalent.

(II) Finite sets:→ A set which consists of a definite number of elements is called finite sets.

For example: $A = \{x : x$ is an integer $-3 < x < 7\}$
Then $A = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$.

III Denumerable Sets:→ If a set X is equinumerous to the set N, then it is said to be denumerable sets.
For example: $X = \{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$, $N \approx X$ with $f(n) = \frac{1}{n}$.

(IV) Countable sets:→ Let a set X is equinumerous to the set N then it is said to be countable sets.
For example: $Z = \{0, 1, -1, 2, -2, \cdots n, -n, \cdots\}$
and $N = \{1, 2, 3, 4, 5, \cdots n, \cdots\}$

(V) Uncountable sets:→ Let a set is neither finite nor countable, it is said to be uncountable sets.

For example: $I = \{x / 0 < x < 1\}$ is uncountable.
Then since $I \subset R$, the uncountability of R will follow.

Q.2: Prove that the set Q of all rational numbers is denumerable.

Sol: proof:→ Let the set of all positive rationals
$$Q^+ = \{x : x = p/q,\ p, q \in N\}$$
and arrange it in the following form

$$\begin{array}{lcccccccc}
A_1 \rightarrow & \frac{1}{1} & \rightarrow \frac{1}{2} & \frac{1}{3} & \rightarrow \frac{1}{4} & \frac{1}{5} & \rightarrow \frac{1}{6} & \cdots & \cdots \\
A_2 & \frac{2}{1} & \boxed{\frac{2}{2}} & \frac{2}{3} & \boxed{\frac{2}{4}} & \frac{2}{5} & \boxed{\frac{2}{6}} & \cdots & \cdots \\
A_3 & \frac{3}{1} & \frac{3}{2} & \boxed{\frac{3}{3}} & \frac{3}{4} & \frac{3}{5} & \boxed{\frac{3}{6}} & \cdots & \cdots \\
A_4 & \frac{4}{1} & \boxed{\frac{4}{2}} & \frac{4}{3} & \boxed{\frac{4}{4}} & \frac{4}{5} & \boxed{\frac{4}{6}} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Here, A_n stands for all rationals with n as numerator. This table contains all positive rationals. Now arranging them in the order shown by the arrowed line in the above table avoiding repetition, we have

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \cdots\cdots$$

Which are to be counted as

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \cdots\cdots$$

Thus the set of positive rationals is countable.

Lecturered By Dr. H.K. Yadav
Deptt. of Mathematics
Marwari College, Darbhanga
B.Sc. part-II(H), paper-III 18-04-2020
Differential Calculus (partial Diff.)

Q.1. If $u = \log(\tan x + \tan y + \tan z)$, show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

Soln: Here, $u = \log(\tan x + \tan y + \tan z)$ — (I)

Differentiating I partially with respect to x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y + \tan z)} \times \sec^2 x = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

~~Similarly, diff. I with respect to y, we get~~

So that is $\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \sec^2 x}{\tan x + \tan y + \tan z}$

$$\Rightarrow \sin 2x \frac{\partial u}{\partial x} = \frac{2\sin x \cos x}{\tan x + \tan y + \tan z} \times \frac{1}{\cos^2 x} = \frac{2\tan x}{\tan x + \tan y + \tan z}$$

Similarly, diff. I w.r. to y, we get

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z}$$

And diff. I w.r. to z, we get

$$\sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z} = \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)}$$

$= 2$ proved

B.Sc - part-II(H), paper-III 18-04-2020.

Q. 2. If $u = \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Soln We have, $u = \tan^{-1}\frac{y}{x} \Rightarrow \tan u = \frac{y}{x}$ —— I

Differentiating I partially w.r. to x, we get

$$\sec^2 u \frac{\partial u}{\partial x} = -\frac{y}{x^2} \Rightarrow (1+\tan^2 u)\frac{\partial u}{\partial x} = -\frac{y}{x^2}$$

$$\Rightarrow \left(1+\frac{y^2}{x^2}\right)\frac{\partial u}{\partial x} = -\frac{y}{x^2} \quad [\text{from I}]$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{y}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \text{ —— II}$$

Also, diff. I partially w. r. to y, we get

$$\sec^2 u \frac{\partial u}{\partial y} = \frac{1}{x} \quad [\text{from I}]$$

$$\Rightarrow \frac{x^2+y^2}{x^2}\frac{\partial u}{\partial y} = \frac{1}{x} \Rightarrow \frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \text{ —— III}$$

Adding II and III, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

proved

Q.3. If $u = \frac{\sin(ct-x)}{x}$, prove that $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x}\right)$

Soln: We have, $u = \frac{\sin(ct-x)}{x}$ $\therefore \frac{\partial u}{\partial t} = \frac{\cos(ct-x)}{x}\cdot c$

and $\frac{\partial^2 u}{\partial t^2} = -\frac{c^2}{x}\sin(ct-x)$ —I

Also, $\frac{\partial u}{\partial x} = -\frac{1}{x^2}\sin(ct-x) - \frac{1}{x}\cos(ct-x)$ —II

and $\frac{\partial^2 u}{\partial x^2} = \frac{1}{x^2}\cos(ct-x) + \frac{2}{x^3}\sin(ct-x) + \frac{1}{x^2}\cos(ct-x) - \frac{1}{x}\sin(ct-x)$ —III

From II and III, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} = -\frac{1}{x}\sin(ct-x) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad [\text{from I}]$$

$$\therefore c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\cdot\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}$$ proved